

THE WULFF CONSTRUCTION FOR CONVEX INTEGRANDS

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ABSTRACT. For any given Wulff shape \mathcal{W} , we can define the unique continuous function $S^n \rightarrow \mathbb{R}_+$ called convex integrand, denoted by $\gamma_{\mathcal{W}}$. In this paper, we show that, for any Wulff shapes \mathcal{W}_1 and \mathcal{W}_2 , the equality $d(\gamma_{\mathcal{W}_1}, \gamma_{\mathcal{W}_2}) = h(\mathcal{W}_1, \mathcal{W}_2)$ holds, where d is the maximum distance of the function space consisting of convex integrands and h is the Pompeiu-Hausdorff distance of the space consisting of Wulff shapes. Moreover, applications of this result are given.

1. INTRODUCTION

Throughout this paper, we let n, S^n and \mathbb{R}_+ be a positive integer, the unit sphere of \mathbb{R}^{n+1} and the set consisting of positive real numbers respectively. Define the set $C^0(S^n, \mathbb{R}_+)$ as follows.

$$C^0(S^n, \mathbb{R}_+) = \{ \gamma \in C^0(S^n, \mathbb{R}_+) \mid \gamma : S^n \rightarrow \mathbb{R}_+ \text{ continuous} \}.$$

For any $\gamma \in C^0(S^n, \mathbb{R}_+)$ and any $\theta \in S^n$, let $\Gamma_{\gamma, \theta}$ be the following half-space, where the dot in the center stands for the scalar product of two vectors $x, \theta \in \mathbb{R}^{n+1}$.

$$\Gamma_{\gamma, \theta} = \{ x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta) \}.$$

The *Wulff shape associated with γ* , denoted by \mathcal{W}_γ , is the following intersection

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma, \theta}.$$

This construction is well-known as the Wulff construction of geometric model for an equilibrium crystal introduced by G. Wulff in [6]. By definition, it is clear that Wulff shape is a convex body containing the origin of \mathbb{R}^{n+1} as an interior point. Conversely, it is known that any convex body containing the origin as an interior point is a Wulff shape associated with an appropriate continuous function ([5]). For details on Wulff shapes, see for example [2, 3, 4, 5].

Given a $\gamma \in C^0(S^n, \mathbb{R}_+)$, set

$$\text{graph}(\gamma) = \{ (\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{\mathbf{0}\} \mid \theta \in S^n \},$$

where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{\mathbf{0}\}$. The mapping $\text{inv} : \mathbb{R}^{n+1} - \{\mathbf{0}\} \rightarrow \mathbb{R}^{n+1} - \{\mathbf{0}\}$, defined as follows, is called the *inversion* with respect to the origin of \mathbb{R}^{n+1} .

$$\text{inv}(\theta, r) = \left(-\theta, \frac{1}{r} \right).$$

Let Γ_γ be the boundary of the convex hull of $\text{inv}(\text{graph}(\gamma))$. If the equality $\Gamma_\gamma = \text{inv}(\text{graph}(\gamma))$ is satisfied, then γ is called a *convex integrand*. The notion of convex

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integrand was firstly introduced by J. Taylor in [5] and it plays a key role for studying Wulff shapes (for details on convex integrands, see for instance [3, 5]). Let $CI(S^n, \mathbb{R}_+)$ be the set consisting of convex integrands.

$$CI(S^n, \mathbb{R}_+) = \{ \gamma \in C^0(S^n, \mathbb{R}_+) \mid \gamma : \text{convex integrand} \}.$$

Let $\mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1})$ be the set consisting of convex bodies containing the origin of \mathbb{R}^{n+1} as an interior point.

$$\begin{aligned} & \mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1}) \\ &= \{ W \subset \mathbb{R}^{n+1} \mid W : \text{convex body and } \mathbf{0} \in \mathbb{R}^{n+1} \text{ is an interior point of } W \}. \end{aligned}$$

Then, the mapping $\mathcal{W} : C^0(S^n, \mathbb{R}_+) \rightarrow \mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1})$, defined by

$$\mathcal{W}(\gamma) = \mathcal{W}_\gamma,$$

is well-defined. The space $C^0(S^n, \mathbb{R}_+)$ (resp., $\mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1})$) is a metric space with respect to the maximum distance (resp., the Pompeiu-Hausdorff distance). For details on the maximum distance and the Pompeiu-Hausdorff distance, see Section 2. It is not difficult to see that the restriction of \mathcal{W} to $CI(S^n, \mathbb{R}_+)$ is continuous and bijective.

Definition 1. A bijective mapping f from a metric space (X, d_X) into a metric space (Y, d_Y) is said to be an *isometry* if the following equality holds for any $x_1, x_2 \in X$.

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

Nextly, let

$$(*) \quad C^0(CI(S^n, \mathbb{R}_+), \mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1}))$$

be the set consisting of continuous mappings from $CI(S^n, \mathbb{R}_+)$ into $\mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1})$ with respect to the maximum distance and the Pompeiu-Hausdorff distance respectively. Then, although it is desirable to have a concrete isometry in (*), since two metric spaces $CI(S^n, \mathbb{R}_+)$ and $\mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1})$ are apparently heterogeneous, even the existence of isometry in (*) seems to be wrapped in mystery.

The purpose of this paper is to solve the existence problem of isometry in (*) by giving a concrete example of isometry. Namely, we show that the Wulff construction \mathcal{W} is actually isometric.

Theorem 1. *The restriction of \mathcal{W} to $CI(S^n, \mathbb{R}_+)$,*

$$\mathcal{W}|_{CI(S^n, \mathbb{R}_+)} : CI(S^n, \mathbb{R}_+) \rightarrow \mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1}),$$

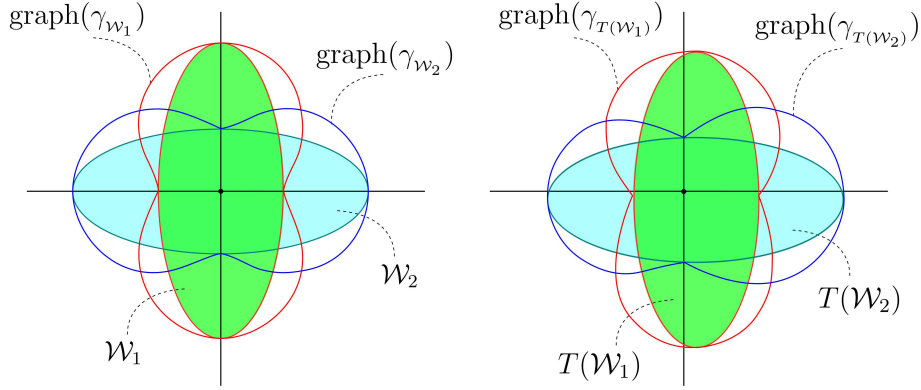
is an isometry.

By Theorem 1, we clearly have the following (see Figure 1).

Corollary 1. *Let $\mathcal{W}_1, \mathcal{W}_2$ be two elements of $\mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1})$ and let $T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be a parallel translation such that the origin $\mathbf{0}$ is an interior point of $T(\mathcal{W}_1) \cap T(\mathcal{W}_2)$. Then, the following equality holds.*

$$d(\gamma_{\mathcal{W}_1}, \gamma_{\mathcal{W}_2}) = d(\gamma_{T(\mathcal{W}_1)}, \gamma_{T(\mathcal{W}_2)}).$$

For any $c \in \mathbb{R}_+$, let $c^* : S^n \rightarrow \mathbb{R}_+$ be the constant function $c^*(S^n) = c$. For any $\mathcal{W} \in \mathcal{H}_{\text{conv}, \mathbf{0}}(\mathbb{R}^{n+1})$, define the function $w^* : S^n \rightarrow \mathbb{R}_+$ by $w^*(\theta) = \min\{c \in \mathbb{R}_+ \mid \mathcal{W} \subset \Gamma_{c^*, \theta}\}$ for any $\theta \in S^n$. Then, notice that $\gamma_{\mathcal{W}} = w^*$. By this observation, the following is clearly obtained as a corollary of Theorem 1.

FIGURE 1. $d(\gamma_{W_1}, \gamma_{W_2}) = d(\gamma_{T(W_1)}, \gamma_{T(W_2)})$.

Corollary 2. *Let W_1, W_2 be two convex bodies such that the intersection $W_1 \cap W_2$ is a convex body. Then, $h(W_1, W_2)$ can be calculated radially from any interior point of $W_1 \cap W_2$. More precisely, the following holds.*

$$h(W_1, W_2) = \max_{\theta \in S^n} |\min\{c \in \mathbb{R}_+ \mid W_1 \subset (x + \Gamma_{c^*, \theta})\} - \min\{c \in \mathbb{R}_+ \mid W_2 \subset (x + \Gamma_{c^*, \theta})\}|,$$

where x is an interior point of $W_1 \cap W_2$.

For given two convex bodies such that their intersection is a convex body, Corollary 2 gives an algorithm that can compute an approximate value of the Pompeiu-Hausdorff distance with high precision.

This paper is organized as follows. In Section 2, the maximum distance and the Pompeiu-Hausdorff distance are reviewed. In Section 3, the proof of Theorem 1 is given.

2. METRIC SPACES $(CI(S^n, \mathbb{R}_+), d)$ AND $(\mathcal{H}_{conv, \mathbf{0}}(\mathbb{R}^{n+1}), h)$.

Definition 2. Let γ_1, γ_2 be two elements of $CI(S^n, \mathbb{R}_+)$. Then, the mapping $d : CI(S^n, \mathbb{R}_+) \times CI(S^n, \mathbb{R}_+) \rightarrow \mathbb{R}$ defined as follows is called *the maximum distance* between γ_1 and γ_2 .

$$d(\gamma_1, \gamma_2) = \max_{\theta \in S^n} |\gamma_1(\theta) - \gamma_2(\theta)|.$$

Definition 3 ([1]). Let (\tilde{X}, \tilde{d}) be a complete metric space.

- (1) Let x (resp., B) be a point of \tilde{X} (resp., a non-empty compact subset of \tilde{X}). Define

$$\tilde{d}(x, B) = \min\{\tilde{d}(x, y) \mid y \in B\}.$$

Then, $\tilde{d}(x, B)$ is called the *distance from the point x to the set B* .

- (2) Let A, B be two non-empty compact subsets of \tilde{X} . Define

$$\tilde{d}(A, B) = \max\{\tilde{d}(x, B) \mid x \in A\}.$$

Then, $\tilde{d}(A, B)$ is called the *distance from the set A to the set B* .

(3) Let A, B be two non-empty compact subsets of \tilde{X} . Define

$$h(A, B) = \max\{\tilde{d}(A, B), \tilde{d}(B, A)\}.$$

Then, $h(A, B)$ is called the *Pompeiu-Hausdorff distance between A and B* .

3. PROOF OF THEOREM 1

Lemma 1. *Given a convex integrand $\gamma \in CI(S^n, \mathbb{R}_+)$ and a positive real number $a \in \mathbb{R}_+$, define the continuous function $\gamma_a : S^n \rightarrow \mathbb{R}_+$ by $\gamma_a(\theta) = \gamma(\theta) + a$ for any $\theta \in S^n$. Then, the following equality holds.*

$$\mathcal{W}_{\gamma, a} = \overline{B(\mathcal{W}_\gamma, a)}.$$

Here, $\overline{B(\mathcal{W}_\gamma, a)}$ means $\bigcup_{P \in \mathcal{W}_\gamma} \overline{B(P, a)} = \bigcup_{P \in \mathcal{W}_\gamma} \{x \in \mathbb{R}^{n+1} \mid \|x - P\| \leq a\}$.

Proof. We first prove the inclusion $\overline{B(\mathcal{W}_\gamma, a)} \subset \mathcal{W}_{\gamma_a}$. Suppose that there exists a point P of the boundary of $\overline{B(\mathcal{W}_\gamma, a)}$ such that P is not included in \mathcal{W}_{γ_a} . Then, by the definition of Wulff shape, there exists a point θ of S^n such that the following sharp inequality holds,

$$\gamma(\theta) + a < P \cdot \theta.$$

Let Q be a point of the boundary of \mathcal{W}_γ such that $d(P, Q) = a$. It is clear that there exists a point Q such that $d(P, Q) \leq a$. Suppose that there exists a point Q of the boundary of \mathcal{W}_γ such that $d(P, Q) < a$. Then, there exists a positive real number ε satisfying $\overline{B(P, \varepsilon)} \subset \overline{B(Q, a)}$. This means $\overline{B(P, \varepsilon)} \subset \overline{B(\mathcal{W}_\gamma, a)}$, which contradicts the fact that P is a point of the boundary of $\overline{B(\mathcal{W}_\gamma, a)}$. Thus, by the sharp inequality $\gamma(\theta) + a < P \cdot \theta$, the following holds.

$$(*) \quad \gamma(\theta) + a < P \cdot \theta = \left(Q + a \frac{P - Q}{\|P - Q\|} \right) \cdot \theta = Q \cdot \theta + \left(a \frac{P - Q}{\|P - Q\|} \right) \cdot \theta.$$

On the other hand, it is clear that the following holds for any θ of S^n .

$$Q \cdot \theta \leq \gamma(\theta) \text{ and } \left(a \frac{P - Q}{\|P - Q\|} \right) \cdot \theta \leq a.$$

Thus, we have the following.

$$Q \cdot \theta + \left(a \frac{P - Q}{\|P - Q\|} \right) \cdot \theta \leq \gamma(\theta) + a.$$

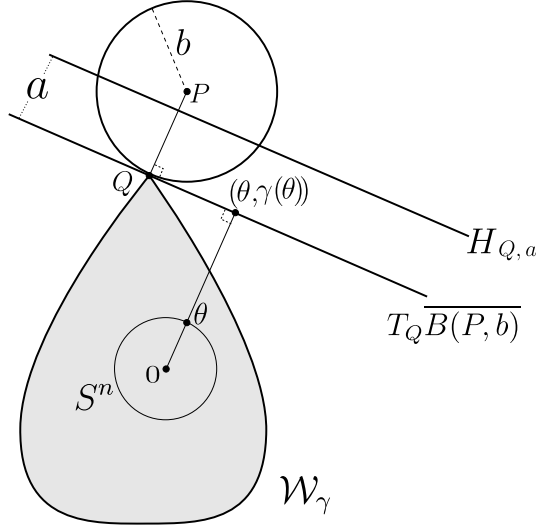
This contradicts (*).

Next, we prove the converse inclusion $\mathcal{W}_{\gamma_a} \subset \overline{B(\mathcal{W}_\gamma, a)}$. Suppose that there exists a point P of \mathcal{W}_{γ_a} such that P is not included in $\overline{B(\mathcal{W}_\gamma, a)}$. Then the intersection of $\overline{B(P, a)}$ and \mathcal{W}_γ is the empty set. Let b be the positive real number such that the following equality holds.

$$\overline{B(P, b)} \cap \partial \mathcal{W}_\gamma = \{Q\}.$$

It is clear that $a < b = \|P - Q\|$. Let $T_Q \overline{B(P, b)}$ be the affine tangent hyperplane to $\overline{B(P, b)}$ at Q (see Figure 2). Since \mathcal{W}_γ is a convex body, it follows that $T_Q \overline{B(P, b)}$ is a support hyperplane to \mathcal{W}_γ at Q . This means that $\mathcal{W}_\gamma \cap T_Q \overline{B(P, b)}$ is a subset of the boundary of \mathcal{W}_γ . Set

$$\theta = \frac{P - Q}{\|P - Q\|}.$$

FIGURE 2. The hyperplanes $T_Q \overline{B(P,b)}$ and $H_{Q,a}$.

Then θ is a point of S^n . Notice that there exists a positive real number λ such that $P - Q = \lambda \theta$. It follows that $Q \cdot \theta = \gamma(\theta)$ and $(\theta, \gamma(\theta)) \in T_Q \overline{B(P,b)}$. By the assumption, the hyperplane

$$H_{Q,a} = \left\{ R \in \mathbb{R}^{n+1} \mid R = M + a \frac{P - Q}{\|P - Q\|}, M \in T_Q \overline{B(P,b)} \right\}$$

does not contain the point Q and the intersection of the segment PQ and $H_{Q,a}$ is not empty (see Figure 2). Thus the following sharp inequality holds,

$$(**) \quad \gamma(\theta) + a = Q \cdot \theta + \left(a \frac{P - Q}{\|P - Q\|} \right) \cdot \theta < Q \cdot \theta + \left(b \frac{P - Q}{\|P - Q\|} \right) \cdot \theta = P \cdot \theta.$$

Since P is a point of \mathcal{W}_{γ_a} , $(**)$ contradicts the inequality $P \cdot \theta \leq \gamma_a(\theta)$. \square

Now we are ready to prove Theorem 1. It is enough to show the following two.

- (1) $h(\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2}) \leq d(\gamma_1, \gamma_2)$ for any $\gamma_1, \gamma_2 \in CI(S^n, \mathbb{R}_+)$.
- (2) $d(\gamma_1, \gamma_2) \leq h(\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2})$ for any $\gamma_1, \gamma_2 \in CI(S^n, \mathbb{R}_+)$.

Suppose that (1) does not hold. Then, there exist two convex integrands γ_1, γ_2 such that the sharp inequality $d(\gamma_1, \gamma_2) < h(\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2})$ holds. Set $d(\gamma_1, \gamma_2) = a > 0$. By Lemma 1, there exists a point P of \mathcal{W}_{γ_1} such that

$$P \notin \overline{B(\mathcal{W}_{\gamma_2}, a)} = \mathcal{W}_{\gamma_2, a}.$$

Then by the definition of Wulff shape, there exists a point θ of S^n such that the inequality $\gamma_2(\theta) + a = \gamma_{2,a}(\theta) < P \cdot \theta$ holds. On the other hand, since P is a point of \mathcal{W}_{γ_1} , we have that $P \cdot \theta \leq \gamma_1(\theta)$ for any θ of S^n . By the assumption, it follows that

$$\gamma_2(\theta) + a < P \cdot \theta \leq \gamma_1(\theta) < \gamma_2(\theta) + a.$$

Thus, we have a contradiction.

Next, we show the inequality (2). Suppose that there exist two convex integrands γ_1, γ_2 such that the sharp inequality $h(\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2}) < d(\gamma_1, \gamma_2)$ holds. Set $h(\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2}) = a > 0$. By the definition of maximum distance, there exists a point θ of S^n satisfying $a < |\gamma_1(\theta) - \gamma_2(\theta)|$. Then, without loss of generality, we may assume that $a < \gamma_1(\theta) - \gamma_2(\theta)$. Notice that for any θ of S^n , there exists a point P_θ of the boundary of \mathcal{W} such that the equality $P_\theta \cdot \theta = \gamma_{\mathcal{W}}(\theta)$ holds. It follows that, for the θ of S^n satisfying $a < \gamma_1(\theta) - \gamma_2(\theta)$, there exists a point P_θ of the boundary of \mathcal{W}_{γ_1} such that the following holds,

$$\gamma_2(\theta) + a < \gamma_1(\theta) = P_\theta \cdot \theta.$$

By Lemma 1, it follows that

$$P_\theta \notin \mathcal{W}_{\gamma_2, a} = \overline{B(\mathcal{W}_{\gamma_2}, a)}.$$

This contradicts the assumption $h(\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2}) = a$. Therefore, the restriction of \mathcal{W} to $CI(S^n, \mathbb{R}_+)$ is an isometry. \square

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